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THE JORDAN PROPERTY OF CREMONA GROUPS AND ESSENTIAL DIMENSION

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Abstract. We use a recent advance in birational geometry to prove new lower bounds on the essential dimension of some finite groups.

1. Introduction

A abstract group Γ is called *Jordan* if there exists an integer j such that every finite subgroup $G \subset \Gamma$ has a normal abelian subgroup A of index $[G : A] \leqslant j$. This definition, due to V. L. Popov [Po11, Po14], was motivated by the classical theorem of Camille Jordan [J1878] which asserts that $\mathrm{GL}_n(k)$ is Jordan, and by a theorem of J.-P. Serre [Se10] which asserts that the Cremona group $\mathrm{Cr}_2(k)$ is also Jordan. Here and throughout this note k denotes a base field of characteristic 0. The Cremona group $\mathrm{Cr}_2 = \mathrm{Bir}(\mathbb{P}^2)$ is the group of birational automorphisms of the projective plane. Serre asked whether the higher Cremona groups $\mathrm{Cr}_n = \mathrm{Bir}(\mathbb{P}^n)$ are Jordan as well. Our starting point is the following remarkable theorem of Y. Prokhorov, C. Shramov and C. Birkar, which asserts that groups of birational isomorphisms of rationally connected varieties of fixed dimension are "uniformly Jordan".

Theorem 1. ([Bi16, Corollary 1.3]) *For every positive integer $n \geqslant 1$ there exists a positive integer $j(n)$ with the following property. Let G be a finite subgroup of the group* $\mathrm{Bir}(X)$ *of birational automorphisms of an n-dimensional rationally connected variety X. Then G has a normal abelian subgroup A such that $[G : A] \leqslant j(n)$.*

Prokhorov and Shramov [PS16] proved this theorem assuming the Borisov-Alexeev-Borisov (BAB) conjecture. The BAB conjecture was subsequently proved by Birkar [Bi16]. In this note we will deduce some consequences of Theorem 1 concerning essential dimension of finite groups.

Let G be a finite group. Recall that the *representation dimension* $\mathrm{rdim}_k(G)$ is the minimal dimension of a faithful representation of G defined over k, i.e., the smallest positive integer r such that G is isomorphic to a subgroup of $\mathrm{GL}_r(k)$. The *essential dimension* $\mathrm{ed}_k(G)$ is the minimal dimension of a faithful linearizable G-variety defined over k. Here by a faithful G-variety we mean an algebraic variety X with a faithful action of G. We say that X is linearizable if there exists a G-equivariant dominant rational map $V \dashrightarrow X$, where V is a vector space with a linear action for G.

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It is clear from these definitions that

$$\mathrm{ed}_k(G) \leqslant \mathrm{rdim}_k(G). \tag{1}$$

We will write $\mathrm{ed}(G)$ and $\mathrm{rdim}(G)$ in place of $\mathrm{ed}_k(G)$ and $\mathrm{rdim}_k(G)$, respectively, when the reference of k is clear from the context. Equality in (1) holds in two interesting cases:

- if G is abelian and k contains a primitive eth root of unity, where e is the exponent of G (see [BR97, Theorem 6.1]), or
- if G is a p-group and k contains a primitive pth root of unity (see [KM08]).

For other finite groups, $\mathrm{ed}(G)$ and $\mathrm{rdim}(G)$ can diverge. Our first main result shows that they do not diverge too far, assuming k contains suitable roots of unity.

Theorem 2. *Let $r(n) = nj(n)$, where $j(n)$ is the Jordan constant from Theorem 1. Suppose G is a finite group of exponent e and the base field k contains a primitive eth root of unity.*

(a) If $\mathrm{ed}_k(G) \leqslant n$, then $\mathrm{rdim}_k(G) \leqslant r(n)$.

(b) Moreover, if $\mathrm{ed}_k(G) \leqslant n$, then G is isomorphic to a finite subgroup of $\mathbb{G}_m^{r(n)} \rtimes \mathrm{S}_{r(n)}$, where the symmetric group $\mathrm{S}_{r(n)}$ acts on $\mathbb{G}_m^{r(n)}$ by permuting the factors.

To place Theorem 2(a) into the context of what is currently known about essential dimension of finite groups, let us assume for simplicity that k is algebraically closed. Let G be a finite group, p be a prime, and $G[p]$ be a Sylow p-subgroup of G. As we mentioned above, $\mathrm{ed}(G[p]) = \mathrm{rdim}(G[p])$ by the Karpenko-Merkurjev theorem [KM08], and $\mathrm{rdim}(G[p])$ can be computed, at least in principle, by the methods of representation theory of finite groups. This way we obtain a lower bound

$$\mathrm{ed}(G) \geqslant \max_p \, \mathrm{rdim}(G[p]). \tag{2}$$

One can then try to prove a matching upper bound by constructing an explicit d-dimensional faithful linearizable G-variety of dimension $\max_p \mathrm{rdim}(G[p])$. In most of the cases where the exact value of $\mathrm{ed}(G)$ is known, it was established using this strategy.

There are, however, finite groups G for which the inequality (2) is strict. All known proofs of stronger lower bounds of the form $\mathrm{ed}(G) > d$ appeal to the classification of finite subgroups of $\mathrm{Bir}(X)$, where X ranges over the d-dimensional unirational (or rationally connected) varieties. Such classifications is available only for $d = 1$ (see [Kl1884, Chapter 1]) and $d = 2$, and the latter is rather complicated; see [DI09]. For $d = 3$ there is only a partial classification (see [Pr12]), and for $d \geqslant 4$ even a partial classification is currently out of reach. Lower bounds of the form $\mathrm{ed}(G) > d$ proved by this method (for suitable finite groups G), can be found

- in [BR97, Theorem 6.2] for $d = 1$,
- in [Se10, Proposition 3.6], [Dun13] for $d = 2$, and
- in [Dun10], [Be14], [Pr17] for $d = 3$.

For an overview, see [Rei10, Section 6]. This paper is in a similar spirit, with Theorem 1 used in place of the above-mentioned classifications.

As a consequence of Theorem 2(a), we will obtain the following.

Theorem 3. *Let $\mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n and H_n be a subgroup of* $\operatorname{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^*$ *for $n = 1, 2, 3, \ldots$. If $\lim_{n\to\infty} |H_n| = \infty$, then $\lim_{n\to\infty} \operatorname{ed}_k((\mathbb{Z}/n\mathbb{Z}) \rtimes H_n) = \infty$ for any field k of characteristic 0.*

In particular, $\lim_{n\to\infty} \operatorname{ed}_{\mathbb{C}}((\mathbb{Z}/n\mathbb{Z}) \rtimes (\mathbb{Z}/n\mathbb{Z})^*) = \infty$. In the case where $n = p$ is a prime, all Sylow subgroups of $(\mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z})^*$ are cyclic, so (2) reduces to the vacuous lower bound

$$\operatorname{ed}_{\mathbb{C}}((\mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z})^*) \geqslant 1.$$

It was not previously known that $\operatorname{ed}((\mathbb{Z}/p\mathbb{Z}) \rtimes (\mathbb{Z}/p\mathbb{Z})^*) > 3$ for any prime p.

2. Proof of Theorem 2

Let $G \to \operatorname{GL}(V)$ be a faithful linear representation of G. By the definition of essential dimension there exists a G-equivariant dominant rational map $V \dashrightarrow X$ such that G acts faithfully on X and $\dim(X) = \operatorname{ed}(G) \leqslant n$. By Theorem 1, there exists a normal abelian subgroup $A \triangleleft G$ such that $[G : A] \leqslant j(n)$.

As we mentioned above, when A is abelian, and k has a primitive eth root of unity, we have $\operatorname{rdim}(A) = \operatorname{ed}(A)$. Since A is a subgroup of G, $\operatorname{ed}(A) \leqslant \operatorname{ed}(G) \leqslant n$. Thus there exists a faithful representation W of A of dimension $d \leqslant n$. The induced representation $V = \operatorname{Ind}_A^G(W)$ of G is clearly faithful, and $\dim(V) = d[G : A] \leqslant nj(n) = r(n)$. Thus $\operatorname{rdim}(G) \leqslant \dim(V) \leqslant r(n)$. This proves (a).

To prove (b), note that in some basis $e_1, \ldots, e_d$ of W, A acts on W by diagonal matrices. Choosing a set of representatives $g_1, \ldots, g_s$ for the cosets of A in G, we see that the vectors g_ie_j form a basis of V, as i ranges from 1 to s and j ranges from 1 to d. The group G permutes the lines $\operatorname{Span}_k(g_ie_j)$ in V. The subgroup of $\operatorname{GL}(V)$ that fixed each of these lines individually is a maximal torus $T = \mathbb{G}_m^{ds}$. The subgroup of $\operatorname{GL}(V)$ that preserves this set of lines is the normalizer N of T in $\operatorname{GL}(V)$, where $N \simeq T \rtimes \mathrm{S}_{ds}$. Thus our faithful representation $G \to \operatorname{GL}(V)$ embeds G in N. Since $s = [G : A] \leqslant j(n)$ and thus $ds \leqslant nj(n) = r(n)$, we can further embed N into $(\mathbb{G}_m)^{r(n)} \rtimes \mathrm{S}_{r(n)}$. □

Remark 4. Without the assumption that k contains a primitive root ζ_e of unity of degree e, our proof of Theorem 2(a) only shows that if there exists a number $a_k(n)$ such that

$$\operatorname{ed}_k(A) \leqslant n \quad \Longrightarrow \quad \operatorname{rdim}_k(A) \leqslant a_k(n)$$

for every finite abelian group A, then

$$\operatorname{ed}_k(G) \leqslant n \quad \Longrightarrow \quad \operatorname{rdim}_k(G) \leqslant a_k(n)j(n).$$

Note that $\operatorname{Bir}(X)(k) \subset \operatorname{Bir}(X)(\overline{k})$, where $\overline{k}$ is the algebraic closure of k, we may assume that the Jordan constant $j(n)$ is the same for k as for $\overline{k}$. On the other hand, $a_k(n)$ may depend on k. Moreover, if k is an arbitrary field of characteristic 0, we do not know whether or not $a(n)$ exists. For example, if p is a prime, then $\operatorname{rdim}_{\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) = p - 1$ is not bounded from above, as p increases, but it is not known whether or not $\operatorname{ed}_{\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z})$ is bounded from above.

Remark 5. Finite subgroups of $\mathbb{G}_m^2 \rtimes S$, for certain small finite groups S play a prominent role in the classification of finite groups of essential dimension 2 (over $\mathbb{C}$), due to A. Duncan; see [Dun13, Theorem 1.1]. Theorem 2(b) suggests that this is not an accident.

Remark 6. In the definition of the Jordan group we could have dropped the assumption that A is normal: Γ is Jordan if and only if there exists an integer $\tilde{j}$ such that every finite subgroup $G \subset \Gamma$ contains an abelian subgroup $A \subset G$ of index $[G : A] < \tilde{j}$. One usually refers to j and $\tilde{j}$ as *the Jordan constant* and *the weak Jordan constant* for Γ, respectively. These constants are related by the inequalities $\tilde{j} \leqslant j \leqslant \tilde{j}^2$; see [PS17, Remark 1.2.2]. Indeed, if G has an abelian subgroup of index $\leqslant i$, then it has a normal abelian subgroup of index $\leqslant i^2$; see [I08, Theorem 1.41].

Now observe that our proof of Theorem 2(a) does not use the fact that A is normal. Thus we could have defined $r(n)$ as $n\tilde{j}(n)$, rather than $nj(n)$, in the statement of Theorem 2(a). This will not make a difference in this paper, but may be helpful if one tries to find an explicit value for $r(n)$ for some (or perhaps, even all) n. The constants $j(n)$ and $\tilde{j}(n)$ are largely mysterious, but some explicit values for $n = 2$ and 3 can be found in [PS17].

Remark 7. It is not true that $[G : Z(G)]$ is bounded from above, as G ranges over the groups of essential dimension $\leqslant n$. Here $Z(G)$ denotes the center of G. For example let D_{2n} be the dihedral group of order $2n$. Then $\mathrm{ed}_{\mathbb{C}}(D_{2n}) = 1$ for every odd integer n (see [BR97, Theorem 6.2]), but $Z(D_{2n}) = 1$, and thus $[D_{2n} : Z(D_{2n})] = 2n$ is unbounded from above, as n ranges over the odd integers.

3. Proof of Theorem 3

Let l be the field obtained from k by adjoining all roots of unity. Since

$$\mathrm{ed}_k(G) \geqslant \mathrm{ed}_l(G)$$

for every finite group G, we may replace k by l and thus assume that k contains all roots of unity. Under this assumption, we can restate Theorem 2(a) as follows. Let $G_1, G_2, \dots$ be a sequence of finite groups.

$$\text{If } \lim_{n\to\infty} \mathrm{rdim}(G_n) = \infty, \text{ then } \lim_{n\to\infty} \mathrm{ed}(G_n) = \infty. \tag{3}$$

The following lemma is elementary; we include a short proof for the sake of completeness.

Lemma 8. *Let $q = p^a$ be a prime power, and $\phi\colon H \to \mathrm{Aut}(\mathbb{Z}/q\mathbb{Z}) = (\mathbb{Z}/q\mathbb{Z})^*$ be a group homomorphism. Then $\mathrm{rdim}((\mathbb{Z}/q\mathbb{Z}) \rtimes_\phi H) \geqslant |\phi(H)|$.*

Proof. Suppose $\rho\colon (\mathbb{Z}/q\mathbb{Z}) \rtimes_\phi H \to \mathrm{GL}(V)$ is a d-dimensional faithful representation. Our goal is to show that $d \geqslant |\phi(H)|$. By our assumption on k, V as a direct sum of 1-dimensional character spaces $V = V_{\chi_1} \oplus \cdots \oplus V_{\chi_d}$ for the cyclic group $\mathbb{Z}/q\mathbb{Z}$, where the characters $\chi_1, \dots, \chi_d$ are permuted by H. Since ρ is faithful, the restriction of one of these characters, say of χ_i, to the unique subgroup of order p in $\mathbb{Z}/q\mathbb{Z}$ is non-trivial. Hence, $\chi_i\colon \mathbb{Z}/q\mathbb{Z} \to k^*$ is faithful. This implies that the H-orbit of χ_i has exactly $|\phi(H)|$ elements. Thus $d \geqslant |\phi(H)|$, as desired. $\square$

We are now ready to proceed with the proof of Theorem 3. Set $G_n = (\mathbb{Z}/n\mathbb{Z}) \rtimes H_n$. By (3), it suffices to show that $\lim_{n\to\infty} \mathrm{rdim}(G_n) = \infty$. In other words, for any positive real number R, we want to show that there are at most finitely many integers $n \geqslant 1$ such that

$$\operatorname{rdim}(G_n) \leqslant R. \tag{4}$$

Write

$$H_n = (\mathbb{Z}/q_1\mathbb{Z})^{a_1} \times \cdots \times (\mathbb{Z}/q_r\mathbb{Z})^{a_r} \tag{5}$$

as a product of cyclic groups, where $q_1, \ldots, q_r$ are distinct prime powers.

Claim: If (4) holds, then (a) $a_i \leqslant R$, and (b) $q_i \leqslant R$, for every $i = 1, \ldots, r$.

Assume for a moment that this claim is established. For a fixed R, there are only finitely many groups of the form $(\mathbb{Z}/q_1\mathbb{Z})^{a_1} \times \cdots \times (\mathbb{Z}/q_r\mathbb{Z})^{a_r}$ satisfying (a) and (b) (recall that $q_1, \ldots, q_r$ are required to be distinct). Since $\lim_{n\to\infty} |H_n| = \infty$, we conclude that the inequality $\operatorname{rdim}(G_n) \leqslant R$ holds for only finitely many integers $n \geqslant 1$, as desired.

It remains to prove the claim. For part (a), note that

$$R \geqslant \operatorname{rdim}(G_n) \geqslant \operatorname{rdim}(H_n) \geqslant \operatorname{rdim}(\mathbb{Z}/q_i\mathbb{Z})^{a_i} = a_i \,.$$

To prove part (b), by symmetry it suffices to show that $q_1 \leqslant R$. Let $n = p_1^{e_1} \ldots p_s^{e_s}$ be the prime decomposition of n and let $\phi_j \colon H_n \to \operatorname{Aut}(\mathbb{Z}/p_j^{e_j}\mathbb{Z})$ be the projection of

$$H_n \subset \operatorname{Aut}(\mathbb{Z}/n\mathbb{Z}) = \operatorname{Aut}(\mathbb{Z}/p_1^{e_1}\mathbb{Z}) \times \cdots \times \operatorname{Aut}(\mathbb{Z}/p_s^{e_s}\mathbb{Z})$$

to the jth factor. Since q_1 is a prime power, at least one of the projections ϕ_j maps the first factor $\mathbb{Z}/q_1\mathbb{Z}$ in (5) isomorphically onto its image. Thus G_n contains a subgroup isomorphic to $(\mathbb{Z}/p_j^{e_j}\mathbb{Z}) \rtimes_{\phi_j} (\mathbb{Z}/q_1\mathbb{Z})$ and by Lemma 8,

$$R \geqslant \operatorname{rdim}(G_n) \geqslant \operatorname{rdim}((\mathbb{Z}/p_j^{e_j}\mathbb{Z}) \rtimes_{\phi_j} (\mathbb{Z}/q_1\mathbb{Z})) \geqslant |\phi_j(\mathbb{Z}/q_1\mathbb{Z})| = q_1.$$

This completes the proof of the claim and thus of Theorem 3. □

Example 9. Fix a prime p. For each $n \geqslant 1$, choose a prime q_n so that $q_n - 1$ is divisible by p^n. There are infinitely many choices of such q_n for each n by Dirichlet's theorem of primes in arithmetic progressions. Embed $\mathbb{Z}/p^n\mathbb{Z}$ into the cyclic group $(\mathbb{Z}/q_n\mathbb{Z})^*$ of order $q_n - 1$ and form the semidirect product $\Gamma_n = (\mathbb{Z}/q_n\mathbb{Z}_n) \rtimes (\mathbb{Z}/p^n\mathbb{Z})$.

It is shown in [BRV18, Example 3.5] that a conjecture of Ledet implies that $\operatorname{ed}_{\mathbb{C}}(\Gamma_n) \geqslant n$. Theorem 3 yields an unconditional proof of a weaker assertion:

$$\lim_{n\to\infty} \operatorname{ed}_{\mathbb{C}}(\Gamma_n) = \infty.$$

As is pointed out in [BRV18], it was not previously known that $\operatorname{ed}_{\mathbb{C}}(\Gamma_n) > 3$ for any n.

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